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# Generalized projective representations for $sl(n+1)$ ☆

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A B S T R A C T

It is well known that $n$-dimensional projective group gives rise to a non-homogeneous representation of the Lie algebra $sl(n+1)$ on the polynomial functions of the projective space. Using Shen's mixed product for Witt algebras (also known as Larsson functor), we generalize the above representation of $sl(n+1)$ to a non-homogeneous representation on the tensor space of any finite-dimensional irreducible $gl(n)$-module with the polynomial space. Moreover, the structure of such a representation is completely determined by employing projection operator techniques and well-known Kostant's characteristic identities for certain matrices with entries in the universal enveloping algebra. In particular, we obtain a new one parameter family of infinite-dimensional irreducible $sl(n+1)$-modules, which are in general not highest-weight type, for any given finite-dimensional irreducible $sl(n)$-module. The results could also be used to study the quantum field theory with the projective group as the symmetry.



## 1. Introduction

A projective transformation on $\mathbb{F}^n$ for a field $\mathbb{F}$ is given by

$$u \mapsto \frac{Au+\vec{b}}{\vec{c}^t u+d} \quad \text{for } u \in \mathbb{F}^n, \tag{1.1}$$

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where all the vectors in $\mathbb{F}^n$ are in column form and

$$\begin{pmatrix} A & \vec{b} \\ \vec{c}^t & d \end{pmatrix} \in GL(n). \tag{1.2}$$

It is well known that a transformation of mapping straight lines to lines must be a projective transformation. The group of projective transformations is the fundamental symmetry of $n$-dimensional projective geometry. Physically, the group with $n=4$ and $\mathbb{F}=\mathbb{R}$ consists of all the transformations of keeping free particles including light signals moving with constant velocities along straight lines (e.g., cf. [12,13]). Based on the embeddings of the Poincaré group and de Sitter group into the projective group with $n=4$ and $\mathbb{F}=\mathbb{R}$, Guo, Wu and Zhou [12,13] proposed three kinds of special relativity.

In this paper, we give a representation-theoretic exploration on the impact of projective transformations. Note that the Lie algebra of $n$-dimensional projective group is spanned by the following differential operations

$$\left\{ \partial_{x_j}, x_i\partial_{x_j}, x_i \sum_{r=1}^{n} x_r \partial_{x_r} \;\middle|\; i, j = 1, 2, \dots, n \right\}, \tag{1.3}$$

which is isomorphic to the special linear Lie algebra $sl(n+1)$. Through the above operators, we obtain a representation of $sl(n+1)$ on the polynomial algebra $\mathbb{F}[x_1, \dots, x_n]$. The non-homogeneity of (1.3) motivates us to generalize the above representation of $sl(n+1)$ to a non-homogeneous representation on the tensor space of any finite-dimensional irreducible $gl(n)$-module with $\mathbb{F}[x_1, \dots, x_n]$ via Shen's mixed product for Witt algebras (cf. [25–27]) (also known as Larsson functor (cf. [17])). It turns out that the structure of such generalized projective representations can be completely determined by employing projection operator techniques (cf. [7]) and well-known Kostant's characteristic identities for certain matrices with entries in the universal enveloping algebra (cf. [16]). In particular, we obtain a new one parameter family of infinite-dimensional irreducible $sl(n+1)$-modules for any given finite-dimensional irreducible $sl(n)$-module.

Denote by $\mathbb{Z}$ the ring of integers and by $\mathbb{N}$ the additive semi-group of nonnegative integers. For any two integers $m$ and $n$, we denote

$$\overline{m,n} = \begin{cases} \{m, m+1, \dots, n\} & \text{if } m \leqslant n, \\ \emptyset & \text{otherwise}. \end{cases} \tag{1.4}$$

Let $\mathcal{A}$ be a commutative associative algebra over $\mathbb{F}$. If $\{D_i \mid i \in \overline{1,n}\}$ is a set of commuting derivations of $\mathcal{A}$, the set of derivations $\mathcal{W}(n) = \{\sum_{i=1}^n a_i D_i \mid a_i \in \mathcal{A}\}$ forms a Lie algebra via the following Lie brackets:

$$\left[ \sum_{i=1}^{n} a_i D_i, \sum_{i=1}^{n} b_i D_i \right] = \sum_{i,j=1}^{n} \big( a_j D_j(b_i) - b_j D_j(a_i) \big) D_i. \tag{1.5}$$

Let $E_{r,s}$ be the square matrix with 1 as its $(r,s)$-entry and 0 as the others. The general linear Lie algebra $gl(n)$ is the Lie algebra of $n \times n$ matrices over $\mathbb{F}$ with a vector space basis $\{E_{i,j} \mid i, j \in \overline{1,n}\}$. For any $gl(n)$-module $V$, we define an action $\pi$ of the Lie algebra $\mathcal{W}(n)$ on $\mathcal{A} \otimes_{\mathbb{F}} V$ by

$$\pi\left( \sum_{i=1}^{n} a_i D_i \right) = \sum_{i,j=1}^{n} D_i(a_j) \otimes E_{i,j} + \sum_{i=1}^{n} a_i D_i \otimes \mathrm{Id}_V. \tag{1.6}$$

Then $\pi$ gives a representation of $\mathcal{W}(n)$ (cf. [25–27]) and the functor from $gl(n)$-modules to $\mathcal{W}(n)$-modules was also later known as Larsson functor (cf. [19]). The structure of the module $\mathcal{A} \otimes_{\mathbb{F}} V$ was determined by Rao [24] when $\mathcal{A} = \mathbb{F}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, $D_i = \partial_{x_i}$ and $V$ is a finite-dimensional irreducible

$gl(n)$-module. Lin and Tan [18] did the similar thing when $\mathcal{A}$ is the algebra of quantum torus. The first author of this paper [30] determined the $\mathcal{W}(n)$-module structure of $\mathcal{A}\otimes_{\mathbb{F}} V$ in the case that $\mathcal{A}$ is a certain semi-group algebra and $D_i$ are locally-finite derivations in [29]. Throughout this paper, we always assume $\operatorname{char}\mathbb{F}=0$.

Take $\mathfrak{h}=\sum_{i=1}^{n}\mathbb{F}E_{i,i}$ as a Cartan subalgebra of $gl(n)$. Assume $\mathcal{A}=\mathbb{F}[x_1,\dots,x_n]$ and let $D_i=\partial_{x_i}$. Embed $sl(n+1)$ into $\mathcal{W}(n)$ via (1.3). Then the space $\mathcal{A}\otimes_{\mathbb{F}} V$ forms an $sl(n+1)$-module with the representation $\pi|_{sl(n+1)}$, which we call a *generalized projective representation*.

Characteristic identities have a long history. The first person to exploit them was Dirac [3], who wrote down what amounts to the characteristic identity for the Lie algebra $so(1,3)$. This particular example is intimately connected with the problem of describing the structure of relativistically invariant wave equations. Such identities have been shown to be powerful tools for the analysis of finite-dimensional representations of Lie groups (cf. [1,2,5]). It has been shown by Kostant [16] (also cf. [10]) that the characteristic identities for semi-simple Lie algebras also hold for infinite-dimensional representations. Moreover, one may construct projection operators analogous to the projection operators of Green [11], and Bracken and Green [2] in finite dimensions. We refer [4,7–9,14,15,19–23,28] for the other works on the identities. Using projection operator techniques and Kostant's characteristic identities, we prove:

**Main Theorem.** *Let $V$ be a finite-dimensional irreducible $gl(n)$-module with highest weight $\mu$. We have the following conclusions:*

(i) *The space $\mathcal{A}\otimes_{\mathbb{F}} V$ is an irreducible $sl(n+1)$-module if and only if*

$$\mu(E_{1,1})+\sum_{j=1}^{n}\mu(E_{j,j})\notin -\mathbb{N}\cup\overline{2,1+\mu(E_{1,1}-E_{2,2})} \tag{1.7}$$

*and*

$$\mu(E_{i,i})+\sum_{j=1}^{n}\mu(E_{j,j})-i\notin\overline{1,\mu(E_{i,i}-E_{i+1,i+1})}\quad \text{for } i\in\overline{2,n-1}. \tag{1.8}$$

(ii) *If one of the conditions in* (1.7) *and* (1.8) *fails, then both $U(sl(n+1))(1\otimes V)$ and $(\mathcal{A}\otimes_{\mathbb{F}} V)/(U(sl(n+1))(1\otimes V))$ are irreducible $sl(n+1)$-modules.*

Note that all $\mu(E_{i,i}-E_{i+1,i+1})$ are nonnegative integers, which determine the corresponding $sl(n)$-module $V$ uniquely. Moreover, the identity matrix in $gl(n)$ are allowed to be any constant map. The above theorem says that given a finite-dimensional irreducible $sl(n)$-module, we can construct a new one-parameter family of explicit irreducible $sl(n+1)$-modules via its projective representation and Shen's mixed product.

A quantum field is an operator value function on certain Hilbert space, which is often a direct sum of infinite-dimensional irreducible modules of certain Lie algebra (group). The Lie algebra of two-dimensional conformal group is exactly the Virasoro algebra, which is infinite-dimensional. The minimal models of two-dimensional conformal field theory were constructed from direct sums of certain infinite-dimensional irreducible modules of the Virasoro algebra, where a distinguished module gives rise to a vertex operator algebra. When $n>2$, the $n$-dimension conformal group is finite-dimensional, whose Lie algebra is exactly isomorphic to $so(n,2)$. It is still unknown what should a higher-dimensional conformal field theory should be. Part of reason is that we are lack of enough knowledge on the infinite-dimensional irreducible $so(n,2)$-modules that are compatible to the natural conformal representations of $so(n,2)$. This motivates us to study explicit infinite-dimensional irreducible modules of finite-dimensional simple Lie algebras by using non-homogeneous polynomial representations and Shen's mixed product for Witt algebras. This paper is the first work in this direction.

As we mentioned earlier, projective groups are important groups in physics. In comparison with the minimal models of two-dimensional conformal field theory, the underlying module of the projective representation of $sl(n+1)$ should be the distinguished module in the possible quantum field theory with the projective group as the symmetry. The other modules would make the theory more substantial.

The paper is organized as follows. In Section 2, we slightly generalize Kostant's characteristic identities and recall some facts about projection operators based on Kostant's work [16] and Gould's works [7,10]. In Section 3, we prove (i) and (ii) in the theorem.

## 2. Characteristic identities and projection operators

In order to keep the paper self-contained, we will first prove certain characteristic identities for $gl(n)$, which will be used to study the irreducibility of the generalized projective representations for $sl(n+1)$. Then we will recall some facts about the projection operators for $gl(n)$, although part of the results in this section has appeared in [7,10,22,16].

### 2.1. Some standard facts for $gl(n)$ and $sl(n)$

Recall that $gl(n)$ is the Lie algebra of $n \times n$ matrices over $\mathbb{F}$ with a basis $\{E_{i,j} \mid i, j \in \overline{1,n}\}$ and the Lie bracket:

$$[E_{i,j}, E_{k,l}] = \delta_{k,j} E_{i,l} - \delta_{i,l} E_{k,j} \quad \text{for } i, j, k, l \in \overline{1,n}. \tag{2.1}$$

Note that $gl(n)$ is reductive with the following decomposition of ideals $gl(n) = sl(n) \oplus \mathbb{F}\mathrm{I}$, where $sl(n)$ is the special linear Lie algebra of matrices with zero trace and $\mathrm{I} = \sum_{i=1}^{n} E_{i,i}$ is the identity matrix, which is central.

Take $\mathfrak{h} = \sum_{i=1}^{n} \mathbb{F} E_{i,i}$ as a Cartan subalgebra of $gl(n)$. If $\lambda \in \mathfrak{h}^*$ is a weight of $gl(n)$, we identify $\lambda$ with the $n$-tuple $\lambda = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_i = \lambda(E_{i,i})$. For $\lambda, \mu \in \mathfrak{h}^*$, we define

$$(\lambda, \mu) = \sum_{i=1}^{n} \lambda_i \mu_i. \tag{2.2}$$

Denote by $\varepsilon_i$ the weight with 1 as its $i$th coordinate and 0 as the others, i.e.

$$\varepsilon_i = (0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0). \tag{2.3}$$

The set $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leqslant i < j \leqslant n\}$ forms a set of positive roots of $gl(n)$. In this case, the half-sum of the positive roots is given by

$$\delta = \frac{1}{2} \sum_{i<j} (\varepsilon_i - \varepsilon_j) = \frac{1}{2} \sum_{i=1}^{n} (n + 1 - 2i) \varepsilon_i. \tag{2.4}$$

Moreover, there exists a one-to-one correspondence between the set of finite-dimensional irreducible $gl(n)$-modules and the set of $n$ tuples $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda_i - \lambda_{i+1} \in \mathbb{N}$ for $i \in \overline{1, n-1}$. Such an $n$ tuple $\lambda$ is called the *highest weight* of the corresponding module which we denote as $V(\lambda)$.

Let $U$ denote the universal enveloping algebra of $gl(n)$ and let $Z$ be the center of $U$. Set

$$\sigma_1 = \mathrm{I}, \qquad \sigma_r = \sum_{i_1, \ldots, i_r = 1}^{n} E_{i_1,i_2} E_{i_2,i_3} \cdots E_{i_{r-1},i_r} E_{i_r,i_1} \quad \text{for } r \in \overline{2,n}. \tag{2.5}$$

Then the center $Z$ of $U$ takes the form (cf. [23])

$$Z = \mathbb{F}[\sigma_1, \dots, \sigma_n]. \tag{2.6}$$

The subspace $\mathfrak{h}_0 = \mathfrak{h} \cap sl(n)$ is a Cartan subalgebra of $sl(n)$ with the standard basis $\{\alpha_i^\vee = E_{i,i} - E_{i+1,i+1} \mid i \in \overline{1, n-1}\}$. Denote the dual vector space of $\mathfrak{h}_0$ by $\mathfrak{h}_0^*$ and the simple root system by $\Delta = \{\alpha_1, \dots, \alpha_n\}$. Let $\omega_1, \omega_2, \dots, \omega_{n-1}$ be the fundamental integral dominant weights in $\mathfrak{h}_0^*$ defined by $\omega_i(\alpha_j^\vee) = \delta_{i,j}$.

Let $V(\psi)$ be the finite-dimensional irreducible $sl(n)$-module with the highest weight $\psi$. We can make $V(\psi)$ as a $gl(n)$-module $V(\psi, b)$ by letting the central element I act as the scalar map $b\,\mathrm{Id}_{V(\psi)}$.

For $\vec{a} = (a_1, \dots, a_{n-1}) \in \mathbb{N}^{n-1}$ and $0 < k \in \mathbb{Z}$, we denote

$$I(\vec{a}, k) = \left\{ (a_1 + c_1 - c_2, a_2 + c_2 - c_3, \dots, a_{n-1} + c_{n-1} - c_n) \;\middle|\; c_i \in \mathbb{N} \text{ such that } \sum_{i=1}^{n} c_i = k \text{ and } c_{s+1} \leqslant a_s \text{ for } s \in \overline{1, n-1} \right\}. \tag{2.7}$$

Moreover, we set

$$\omega_{\vec{a}} = \sum_{i=1}^{n-1} a_i \omega_i \quad \text{for } \vec{a} \in \mathbb{N}^{n-1}. \tag{2.8}$$

**Lemma 2.1.1.** *(See e.g., Proposition 15.25 in [6].) For any $\vec{a} \in \mathbb{N}^{n-1}$, the tensor product of $sl(n)$-module $V(\omega_{\vec{a}})$ with $V(k\omega_1)$ decomposes into a direct sum:*

$$V(\omega_{\vec{a}}) \otimes_{\mathbb{F}} V(k\omega_1) = \bigoplus_{\vec{b} \in I(\vec{a}, k)} V(\omega_{\vec{b}}). \tag{2.9}$$

Denote

$$\underline{k} = (k_1, k_2, \dots, k_n) \in \mathbb{N}^n, \qquad |\underline{k}| = \sum_{i=1}^{n} k_i,$$

$$I(\mu, j) = \left\{ \underline{c} = (c_1, \dots, c_n) \;\middle|\; c_i \in \mathbb{N},\ |\underline{c}| = j,\ c_{s+1} \leqslant \mu_s - \mu_{s+1} \text{ for } s \in \overline{1, n-1} \right\}. \tag{2.10}$$

**Lemma 2.1.2.** *The tensor product of $gl(n)$-module $V(\mu)$ with $V(k\varepsilon_1)$ decomposes into a direct sum:*

$$V(\mu) \otimes_{\mathbb{F}} V(k\varepsilon_1) = \bigoplus_{\underline{c} \in I(\mu, k)} V(\mu + \underline{c}). \tag{2.11}$$

**Proof.** Let $\Pi$ be the weight set of $gl(n)$-module $V(\psi, b)$. Assume $\psi = \sum_{i=1}^{n-1} a_i \omega_i$, $\nu = (\nu_1, \dots, \nu_n) \in \Pi$ and $(\nu_1 - \nu_2, \dots, \nu_{n-1} - \nu_n) = \psi - \sum_{i=1}^{n-1} k_i \alpha_i$. Then we obtain

$$\nu_1 = \sum_{i=1}^{n-1} a_i + \frac{b - \sum_{i=1}^{n-1} i a_i}{n} - k_1, \qquad \nu_n = \frac{b - \sum_{i=1}^{n-1} i a_i}{n} + k_{n-1},$$

$$\nu_j = \sum_{i=j}^{n-1} a_i + \frac{b - \sum_{i=1}^{n-1} ia_i}{n} + k_{j-1} - k_j, \quad j \in \overline{2, n-1}. \tag{2.12}$$

By (2.12) and Lemma 2.1.1, we get (2.11). □

*2.2. Characteristic identities and projection operators for gl(n)*

Now we will introduce the characteristic identities for $gl(n)$. We know that the universal enveloping algebra $U$ of $gl(n)$ can be imbedded into $U \otimes U$ by the associative algebra homomorphism $d : U \to U \otimes U$ determined by

$$d(u) = u \otimes 1 + 1 \otimes u \quad \text{for } u \in gl(n). \tag{2.13}$$

Let $V(\lambda)$ be a fixed finite-dimensional $gl(n)$-module with highest weight $\lambda$ and let $\pi_\lambda$ be the corresponding representation. Kostant [16] considered the map

$$\begin{aligned} \partial : U &\to \big(\operatorname{End} V(\lambda)\big) \otimes_{\mathbb{F}} U, \\ u &\mapsto \operatorname{Id} \otimes u + \pi_\lambda(u) \otimes 1 \end{aligned} \tag{2.14}$$

for $u \in gl(n)$ and extended $\partial$ to an associative algebra homomorphism from $U$ to $(\operatorname{End} V(\lambda)) \otimes_{\mathbb{F}} U$. For $z \in Z$, we denote

$$\tilde{z} = -\frac{1}{2}\big[\partial(z) - \pi_\lambda(z) \otimes 1 - \operatorname{Id} \otimes z\big], \tag{2.15}$$

which may be viewed as an $m \times m$ ($m = \dim V(\lambda)$) matrix with entries in $U$.

Denote by $\chi_\zeta$ the central character of a highest weight $gl(n)$-module with highest weight $\zeta$. Suppose now that $W$ is another $gl(n)$-module admitting the central character $\chi_\mu$ and $\pi_\mu$ is the corresponding representation. We extend $\pi_\mu$ to an algebra homomorphism

$$\begin{aligned} \tilde{\pi}_\mu : \big(\operatorname{End} V(\lambda)\big) \otimes_{\mathbb{F}} U &\to \big(\operatorname{End} V(\lambda)\big) \otimes_{\mathbb{F}} (\operatorname{End} W), \\ \sum_i \rho_i \otimes u_i &\mapsto \sum_i \rho_i \otimes \pi_\mu(u_i), \end{aligned} \tag{2.16}$$

where $\rho_i \in \operatorname{End} V(\lambda)$ and $u_i \in U$. In particular, we have

$$\tilde{\pi}_\mu(\tilde{z}) = -\frac{1}{2}\big[(\pi_\lambda \otimes \pi_\mu)(z) - \pi_\lambda(z) \otimes 1 - \operatorname{Id} \otimes \pi_\mu(z)\big] \tag{2.17}$$

for $z \in Z$. Clearly, $\tilde{\pi}_\mu(\tilde{z})$ is a linear operator on $V(\lambda) \otimes_{\mathbb{F}} W$ which may be viewed as an $m \times m$ matrix with entries from $\operatorname{End} W$ under a basis of $V(\lambda)$. For $\nu \in \mathfrak{h}^*$, we define

$$f_\nu = -\frac{1}{2}(\chi_{\mu+\nu} - \chi_\lambda - \chi_\mu). \tag{2.18}$$

Denote by $\Pi_\lambda$ the weight set of $V(\lambda)$.

**Lemma 2.2.1.** *(See [16,11,22].) On the space $W$, the matrix $\tilde{z}$ satisfies the following characteristic identity*:

$$\prod_{\nu\in\Pi_\lambda}\big(\tilde{z}-f_\nu(z)\big)=0\quad \text{for } z\in Z. \tag{2.19}$$

By varying the module $V(\lambda)$ and the central element $z$, we obtain a series of characteristic identities. In particular, the following characteristic identity will be used in the proof of the Main Theorem:

**Corollary 2.2.2.** *Take $V(\lambda)$ to be the dual module of $gl(n)$-module $V(2,1,\dots,1)$. Then the matrix $\tilde{\sigma}_2$ satisfies the following characteristic identity on $W$*:

$$\prod_{i=1}^{n}(\tilde{\sigma}_2-m_i)=0,\quad m_i=\frac{1}{2}(\lambda,\lambda+2\delta)-\frac{1}{2}\big(\lambda_i,\lambda_i+2(\mu+\delta)\big), \tag{2.20}$$

*where* $\lambda_i=(-1,\dots,\overset{i}{-2},\dots,-1)$, $i\in\overline{1,n}$.

**Proof.** Note that the module $V(2,1,\dots,1)$ has a basis $\{e_1,\dots,e_n\}$ such that

$$E_{i,i}(e_k)=(1+\delta_{i,k})e_k,\qquad E_{i,j}(e_k)=\delta_{j,k}e_i\quad (i\neq j). \tag{2.21}$$

Let $\pi^*$ be the dual module of $gl(n,\mathbb{F})$-module $V(2,1,\dots,1)$. Then

$$\pi^*(E_{i,i})=-(E_{i,i}+\mathrm{I}),\qquad \pi^*(E_{i,j})=-E_{j,i}\quad (i\neq j). \tag{2.22}$$

Obviously, the representation $\pi^*$ is $n$-dimensional and all its weights

$$\big\{\lambda_1=(-2,-1,\dots,-1),\dots,\lambda_i=(-1,\dots,\overset{i}{-2},\dots,-1),\dots,\lambda_n=(-1,\dots,-1,-2)\big\} \tag{2.23}$$

occur with multiplicity one. The matrix $\tilde{\sigma}_2$ in this case is given by

$$\tilde{\sigma}_2=-\sum_{i,j}^{n}\pi^*(E_{j,i})E_{i,j}. \tag{2.24}$$

This is the matrix

$$\tilde{\sigma}_2=\begin{bmatrix} \mathrm{I}+E_{1,1} & E_{1,2} & \cdots & E_{1,n}\\ E_{2,1} & \mathrm{I}+E_{2,2} & \cdots & E_{2,n}\\ \vdots & \vdots & \cdots & \vdots\\ E_{n,1} & E_{n,2} & \cdots & \mathrm{I}+E_{n,n}\end{bmatrix}. \tag{2.25}$$

By Lemma 2.2.1, the operator $\tilde{\sigma}_2$ satisfies the characteristic identity on $W$:

$$\prod_{i=1}^{n}(\tilde{\sigma}_2-m_i)=0,\quad m_i=\frac{1}{2}(\lambda,\lambda+2\delta)-\frac{1}{2}\big(\lambda_i,\lambda_i+2(\mu+\delta)\big),\quad i\in\overline{1,n}.\qquad \square \tag{2.26}$$

In the rest of this section, we will recall some facts about projection operators appeared in [8]. Take $gl(n)$-module $V(\lambda)=V(\varepsilon_1)^*$ (resp. $V(\varepsilon_1)$). Then the corresponding matrices $\tilde{\sigma}_2$ are

$$M = (E_{i,j})_{i,j=1}^{n}, \qquad \tilde{M} = -M^T \in U\big(gl(n)\big), \tag{2.27}$$

respectively. On the space $W$, the matrices $M$ and $\tilde{M}$ satisfy the following characteristic identities:

$$\prod_{i=1}^{n}(M - d_i) = 0, \quad d_i = \mu_i + n - i, \tag{2.28}$$

$$\prod_{i=1}^{n}(\tilde{M} - \tilde{d}_i) = 0, \quad \tilde{d}_i = n - 1 - d_i, \quad i \in \overline{1,n}. \tag{2.29}$$

For $r \in \overline{1,n}$,

$$P_r = \prod_{r \neq l \in \overline{1,n}} \left(\frac{M - d_l}{d_r - d_l}\right), \qquad \tilde{P}_r = \prod_{r \neq l \in \overline{1,n}} \left(\frac{\tilde{M} - \tilde{d}_l}{\tilde{d}_r - \tilde{d}_l}\right) \tag{2.30}$$

are called *projection operators*, which project the tensor product space $V(\varepsilon_1)^* \otimes_{\mathbb{F}} W$ (resp. $V(\varepsilon_1) \otimes_{\mathbb{F}} W$) onto the irreducible module $W_r = P_r(V(\varepsilon_1)^* \otimes_{\mathbb{F}} W)$ (resp. $\tilde{W}_r = \tilde{P}_r(V(\varepsilon_1) \otimes_{\mathbb{F}} W)$) with central character $\chi_{\nu-\varepsilon_r}$ (resp. $\chi_{\nu+\varepsilon_r}$).

## 3. Generalized projective representations

In this section, we will give the detailed construction of generalized projective representations for the special linear Lie algebra $sl(n+1)$ and study their irreducibility.

### 3.1. Construction of the representations

Let $\mathcal{A} = \mathbb{F}[x_1, \ldots, x_n]$. The operator $p_i = x_i \sum_{i=1}^{n} x_i \partial_{x_i}$ is called *pseudo-translation operator* on $\mathcal{A}$ in physics. Note that the Lie algebra of $n$-dimensional projective group

$$L_{n+1} = \sum_{i,j=1}^{n} \mathbb{F} x_i \partial_{x_j} + \sum_{i=1}^{n} \mathbb{F} \partial_{x_i} + \sum_{i=1}^{n} \mathbb{F} p_i, \tag{3.1}$$

forms a Lie subalgebra of Witt algebra

$$\mathcal{W}(n) = \left\{ \sum_{i=1}^{n} f_i \partial_{x_i} \;\middle|\; f_i \in \mathcal{A} \right\}. \tag{3.2}$$

Moreover, we have the following Lie brackets:

$$[\partial_{x_j}, p_i] = \begin{cases} x_i \partial_{x_j}, & i \neq j, \\ \sum_{i=1}^{n} x_i \partial_{x_i} + x_i \partial_{x_i}, & i = j, \end{cases} \tag{3.3}$$

$$[x_i \partial_{x_j}, p_k] = \delta_{j,k} p_i, \qquad [x_i \partial_{x_j}, \partial_{x_k}] = -\delta_{i,k} \partial_{x_j}, \qquad [x_i \partial_{x_j}, x_k \partial_{x_l}] = \delta_{j,k} x_i \partial_{x_l} - \delta_{i,l} x_k \partial_{x_j}. \tag{3.4}$$

To abbreviate, we denote

$$P = \sum_{i=1}^{n} \mathbb{F} p_i, \qquad S = \sum_{i=1}^{n} \mathbb{F} \partial_{x_i}, \qquad \bar{L}_n = \sum_{i,j=1}^{n} \mathbb{F} x_i \partial_{x_j}, \qquad \bar{L}'_n = [\bar{L}_n, \bar{L}_n]. \tag{3.5}$$

Then

$$L_{n+1} = P \oplus S \oplus \bar{L}_n, \qquad [P, P] = \{0\}, \qquad [S, S] = \{0\}. \tag{3.6}$$

Moreover, $\bar{L}_n$ (resp. $\bar{L}'_n$) is isomorphic to $gl(n)$ (resp. $sl(n)$). It is easy to verify:

**Lemma 3.1.1.** *The special linear Lie algebra $sl(n+1)$ is isomorphic to $L_{n+1}$ with the following identification of Chevalley generators*:

$$h_i = x_i \partial_{x_i} - x_{i+1} \partial_{x_{i+1}}, \qquad h_n = \sum_{i=1}^{n} x_i \partial_{x_i} + x_n \partial_{x_n}, \tag{3.7}$$

$$e_i = x_i \partial_{x_{i+1}}, \qquad f_i = x_{i+1} \partial_{x_i}, \qquad e_n = p_n, \qquad f_n = -\partial_{x_n}, \tag{3.8}$$

*for $i \in \overline{1, n-1}$.*

For any finite-dimensional $gl(n)$-module $V$ with highest weight $\mu = (\mu_1, \ldots, \mu_n)$, we define an action $\pi$ of Witt Lie algebra $\mathcal{W}(n)$ on $\mathcal{A} \otimes_{\mathbb{F}} V$ by

$$\pi\left(\sum_{i=1}^{n} a_i D_i\right) = \sum_{i,j=1}^{n} D_i(a_j) \otimes E_{i,j} + \sum_{i=1}^{n} a_i D_i \otimes \mathrm{Id}_V . \tag{3.9}$$

Embed $sl(n+1)$ into $\mathcal{W}(n)$ by (3.7) and (3.8). The space $\mathcal{A} \otimes_{\mathbb{F}} V$ forms an $sl(n+1)$-module with the representation $\pi|_{sl(n+1)}$, which we call a *generalized projective representation* of $sl(n+1)$.

It follows from (3.7)–(3.9) that the explicit $L_{n+1}$-module $\mathcal{A} \otimes_{\mathbb{F}} V$ structure is given by:

$$(x_i \partial_{x_j}).(f \otimes v) = (x_i \partial_{x_j}) f \otimes v + f \otimes E_{i,j}.v, \tag{3.10}$$

$$\partial_{x_i}.(f \otimes v) = \partial_{x_i}(f) \otimes v, \tag{3.11}$$

$$p_i.(f \otimes v) = p_i(f) \otimes v + x_i f \otimes \sum_{i=1}^{n} E_{i,i}.v + \sum_{j=1}^{n} x_j f \otimes E_{i,j}.v \tag{3.12}$$

for $i, j \in \overline{1, n}$, where $f \in \mathcal{A}$ and $v \in V$.

### 3.2. Irreducibility criteria for generalized projective representations

In this section, we will give two irreducibility criteria for $L_{n+1}$-module $\mathcal{A} \otimes_{\mathbb{F}} V$ (cf. Lemmas 3.2.2 and 3.2.3).

Denote

$$x^{\underline{c}} = x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n} \quad \text{for } \underline{c} = (c_1, \ldots, c_n) \in \mathbb{N}^n. \tag{3.13}$$

Recall that $\Pi_\mu$ denotes the weight set of $V$. By (3.10), we have

$$(x_i \partial_{x_i}).\left(x^{\underline{c}} \otimes v_\nu\right) = (c_i + \nu_i) x^{\underline{c}} \otimes v_\nu, \tag{3.14}$$

$$\left(\sum_{i=1}^{n} x_i \partial_{x_i}\right).\left(x^{\underline{c}} \otimes v_\nu\right) = \left(|\underline{c}| + \sum_{i=1}^{n} \nu_i\right) x^{\underline{c}} \otimes v_\nu, \tag{3.15}$$

for any $x^{\underline{c}} \in \mathcal{A}$ and $\nu \in \Pi_\mu$. Assume that $\{v_1, \ldots, v_\ell\}$ is a basis of $V$. For $1 \leqslant k \in \mathbb{N}$, we set

$$U(P)(1 \otimes_{\mathbb{F}} V)_{\langle k\rangle} = \mathrm{Span}_{\mathbb{F}}\{p_{i_1}p_{i_2}\cdots p_{i_k}(1 \otimes v_j) \mid 1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \leqslant n,\ j \in \overline{1,d}\}, \tag{3.16}$$

$$(\mathcal{A} \otimes_{\mathbb{F}} V)_{\langle k\rangle} = \mathrm{Span}_{\mathbb{F}}\{x^{\underline{c}} \otimes v_j \mid x^{\underline{c}} \in \mathcal{A},\ |\underline{c}| = k,\ j \in \overline{1,l}\}. \tag{3.17}$$

Then (3.14) and (3.15) imply that

$$\mathcal{A} \otimes_{\mathbb{F}} V = \bigoplus_{k\in\mathbb{N}} (\mathcal{A} \otimes_{\mathbb{F}} V)_{\langle k\rangle}; \qquad U(P)(1 \otimes_{\mathbb{F}} V) = \bigoplus_{k\in\mathbb{N}} \big(U(P)(1 \otimes V)\big)_{\langle k\rangle}. \tag{3.18}$$

**Lemma 3.2.1.** *The vector space $U(P)(1 \otimes_{\mathbb{F}} V)$ is an irreducible $L_{n+1}$-submodule of $\mathcal{A} \otimes_{\mathbb{F}} V$, where $U(P)$ denote the universal enveloping algebra of abelian Lie algebra $P$.*

**Proof.** Denote

$$\triangle^k_{i,j} = \begin{cases} x_j\partial_{x_i} & \text{if } i \neq j, \\ \sum_{i=1}^n x_i\partial_{x_i} + x_i\partial_{x_i} + k - 1 & \text{if } i = j. \end{cases} \tag{3.19}$$

By induction, we can easily verify the following formula:

$$\partial_{x_i}.p_{i_1}p_{i_2}\cdots p_{i_k}(1 \otimes v_j) = \sum_{s=1}^n p_{i_1}p_{i_2}\cdots \hat{p}_{i_s}\cdots p_{i_k}\triangle^k_{i,i_s}(1 \otimes v_j), \tag{3.20}$$

$$\begin{aligned}(x_i\partial_{x_l}).p_{i_1}p_{i_2}\cdots p_{i_k}(1 \otimes v_j) = {}& \sum_{s=1}^n \delta_{l,i_s} p_{i_1}p_{i_2}\cdots \hat{p}_{i_s}\cdots p_{i_k}p_i(1 \otimes v_j) \\ & + p_{i_1}p_{i_2}\cdots p_{i_k}(1 \otimes E_{i,l}.v_j),\end{aligned} \tag{3.21}$$

$$p_i.p_{i_1}p_{i_2}\cdots p_{i_k}(1 \otimes v_j) = p_ip_{i_1}p_{i_2}\cdots p_{i_k}(1 \otimes v_j). \tag{3.22}$$

So (3.20)–(3.22) imply that $U(P)(1 \otimes_{\mathbb{F}} V)$ is an $L_{n+1}$-submodule of $\mathcal{A} \otimes_{\mathbb{F}} V$. Furthermore, it is easy to verify that

$$\bigcap_{i=1}^n \mathrm{Ker}\,\partial_{x_i}|_{(\mathcal{A}\otimes_{\mathbb{F}}V)_{\langle k\rangle}} = \{0\} \quad \text{for any } 1 \leqslant k \in \mathbb{N}. \tag{3.23}$$

Thus any non-trivial submodule of $U(P)(1 \otimes_{\mathbb{F}} V)$ must contain $1 \otimes V$. The irreducibility of $U(P)(1 \otimes_{\mathbb{F}} V)$ follows. $\square$

In the rest of this section, we will investigate the condition for $\mathcal{A} \otimes_{\mathbb{F}} V = U(P)(1 \otimes_{\mathbb{F}} V)$.

It is obvious that $\mathcal{A} \otimes_{\mathbb{F}} V = U(P)(1 \otimes_{\mathbb{F}} V)$ if and only if $(\mathcal{A} \otimes_{\mathbb{F}} V)_{\langle k\rangle} = (U(P)(1 \otimes_{\mathbb{F}} V))_{\langle k\rangle}$ holds for any $k \in \mathbb{N}$.

Suppose that $B_i$ is an ordered basis for $(\mathcal{A} \otimes_{\mathbb{F}} V)_{\langle i\rangle}$. Denote by $P^j_{i+1,i}$ the matrix of the linear map $p_j|_{(\mathcal{A}\otimes_{\mathbb{F}}V)_{\langle i\rangle}}$ (cf. (3.12)) with respect to the bases $B_i$ and $B_{i+1}$. For any $0 < j \in \mathbb{Z}$, we denote

$$\Gamma_j = \{\hat{i} = (i_1, i_2, \ldots, i_j) \mid i_s \in \overline{1,n};\ i_1 \leqslant i_2 \leqslant \cdots \leqslant i_j\}. \tag{3.24}$$

Set

$$P_{\hat{i}} = P^{i_1}_{j,j-1} P^{i_2}_{j-1,j-2} \cdots P^{i_j}_{1,0}. \tag{3.25}$$

We order

$$\Gamma_j = \{\hat{k}^1, \hat{k}^2, \dots, \hat{k}^{\ell_j}\} \tag{3.26}$$

lexically. In particular,

$$\hat{k}^1 = (1, 1, \dots, 1), \qquad \hat{k}^{\ell_j} = (n, n, \dots, n). \tag{3.27}$$

Then $(U(P)(1 \otimes_{\mathbb{F}} V))_{\langle j \rangle}$ is, as a vector space, isomorphic to the column space of the $\ell \ell_j \times \ell \ell_j$ matrix

$$M_j = [P_{\hat{k}^1}, P_{\hat{k}^2}, \dots, P_{\hat{k}^{\ell_j}}] \tag{3.28}$$

(recall that $\ell = \dim V$). Denote

$$I_1 = \{1\} \cup \{i \in \overline{2,n} \mid \mu_{i-1} - \mu_i \geqslant 1\}. \tag{3.29}$$

By means of the characteristic identity in Corollary 2.2.2, we get the following necessary but not sufficient condition for the irreducibility of $L_{n+1}$-module $\mathcal{A} \otimes_{\mathbb{F}} V$:

**Lemma 3.2.2.** *For any $1 \leqslant k \in \mathbb{N}$. If $\mu_i + |\mu| - i + s \neq 0$ for any $s \in \overline{1,k}$, $i \in I_1$, then $L_{n+1}$-module $\mathcal{A} \otimes_{\mathbb{F}} V$ is irreducible.*

**Proof.** The lemma is based on the following result:

**Claim.** *For any $1 \leqslant k \in \mathbb{N}$, if $\mu_i + |\mu| - i + s \neq 0$, $\forall s \in \overline{1,k}$, $i \in I_1$, then $(\mathcal{A} \otimes_{\mathbb{F}} V)_{\langle k \rangle} = (U(P)(1 \otimes_{\mathbb{F}} V))_{\langle k \rangle}$ holds.*

We will prove this claim by induction on $k$. For $k = 1$, $(U(P)(1 \otimes_{\mathbb{F}} V))_{\langle 1 \rangle}$ is isomorphic as vector space to the column space of $\ell n \times \ell n$ matrix

$$\left[P^1_{1,0}, P^2_{1,0}, \dots, P^n_{1,0}\right] \quad \text{with } P^k_{1,0} = \begin{bmatrix} E_{k,1}|_V \\ E_{k,2}|_V \\ \vdots \\ (\mathrm{I} + E_{k,k})|_V \\ \vdots \\ E_{k,n}|_V \end{bmatrix}, \tag{3.30}$$

which is exactly $\tilde{\sigma}_2|_V$ (cf. (2.25)). By Corollary 2.2.2, the matrix $\tilde{\sigma}_2|_V$ is diagonalizable and it has full rank if and only if all its eigenvalues are not zero, i.e.

$$m_i = \frac{1}{2}(\lambda, \lambda + 2\delta) - \frac{1}{2}\big(\lambda_i, \lambda_i + 2(\mu + \delta)\big) = \mu_i + |\mu| - i + 1 \neq 0 \tag{3.31}$$

for any $i \in I_1$.

Now suppose that the lemma holds for $k = \iota - 1$. Assume $k = \iota$. Note that the eigenvalues of the matrix

$$\begin{bmatrix} (\mathrm{I}+E_{1,1}+s-1)|_V & E_{1,2}|_V & \cdots & E_{1,n}|_V \\ E_{2,1}|_V & (\mathrm{I}+E_{2,2}+s-1)|_V & \cdots & E_{2,n}|_V \\ \vdots & \vdots & \cdots & \vdots \\ E_{n,1}|_V & E_{n,2}|_V & \cdots & (\mathrm{I}+E_{n,n}+s-1)|_V \end{bmatrix} \tag{3.32}$$

are $\mu_i + \sum_{j=1}^{n} \mu_j - i + s$ $(i \in I_1)$ by (3.31). Thus it is invertible if and only if $\mu_i + |\mu| - i + s \neq 0$, $\forall i \in I_1$.

Observe that $(\mathcal{A} \otimes_{\mathbb{F}} V)_{\langle k \rangle} = (U(P)(1 \otimes_{\mathbb{F}} V))_{\langle k \rangle}$ if and only if the vectors

$$\{p_{i_1} \cdots p_{i_k}(1 \otimes v_j) \mid 1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant n,\ j \in \overline{1,\ell}\} \tag{3.33}$$

are linearly independent. Equivalently, the corresponding $\ell\ell_k \times \ell\ell_k$ system of homogeneous linear equations

$$M_k X = 0 \tag{3.34}$$

has only zero solution by (3.31).

Suppose

$$\sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \leqslant n,\ j \in \overline{1,d}} a^j_{i_1,i_2,\ldots,i_k} p_{i_1} p_{i_2} \cdots p_{i_k}(1 \otimes v_j) = 0, \quad a^j_{i_1,i_2,\ldots,i_k} \in \mathbb{F}. \tag{3.35}$$

It can be written as

$$\sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \leqslant n} p_{i_1} p_{i_2} \cdots p_{i_k}(1 \otimes w_{i_1,i_2,\ldots,i_k}) = 0, \quad w_{i_1,i_2,\ldots,i_k} \in V. \tag{3.36}$$

Then

$$\partial_{x_l} . \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \leqslant n} p_{i_1} p_{i_2} \cdots p_{i_k}(1 \otimes w_{i_1,i_2,\ldots,i_k}) = 0, \quad \forall l \in \overline{1,n}. \tag{3.37}$$

Equivalently,

$$\sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \leqslant n} \sum_{s=1}^{n} p_{i_1} p_{i_2} \cdots \hat{p}_{i_s} \cdots p_{i_k} \triangle^k_{l,i_s}(1 \otimes w_{i_1,i_2,\ldots,i_k}) = 0 \tag{3.38}$$

by (3.20). Moreover, it can be written as the form

$$\sum_{1 \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_{k-1} \leqslant n} p_{j_1} p_{j_2} \cdots p_{j_{k-1}}(1 \otimes u_{j_1,j_2,\ldots,j_{k-1}}) = 0, \tag{3.39}$$

where

$$
\begin{aligned}
&1 \otimes u_{j_1,j_2,\ldots,j_{k-1}} \\
&\quad = \sum_{i=1}^{j_1} \triangle_{l,i}^{k}(1 \otimes w_{i,j_1,j_2,\ldots,j_{k-1}}) + \sum_{i=j_1}^{j_2} \triangle_{l,i}^{k}(1 \otimes w_{j_1,i,j_2,\ldots,j_{k-1}}) + \cdots \\
&\qquad + \sum_{i=j_{s-1}}^{j_s} \triangle_{l,i}^{k}(1 \otimes w_{j_1,j_2,\ldots,j_{s-1},i,j_s,\ldots j_{k-1}}) + \cdots \\
&\qquad + \sum_{i=j_{k-1}}^{n} \triangle_{l,i}^{k}(1 \otimes w_{j_1,j_2,\ldots,j_{k-1},i}) = \text{the } l\text{-th row of matrix } N_{k-1}\varpi_n
\end{aligned} \tag{3.40}
$$

with

$$
N_{k-1} = \begin{bmatrix} \sum_{i=1}^{n} x_i\partial_{x_i} + x_1\partial_{x_1} + k - 1 & x_2\partial_{x_1} & \cdots & x_n\partial_{x_1} \\ x_1\partial_{x_2} & \sum_{i=1}^{n} x_i\partial_{x_i} + x_2\partial_{x_2} + k - 1 & \cdots & x_n\partial_{x_2} \\ \vdots & \vdots & \cdots & \vdots \\ x_1\partial_{x_n} & x_2\partial_{x_n} & \cdots & \sum_{i=1}^{n} x_i\partial_{x_i} + x_n\partial_{x_n} + k - 1 \end{bmatrix} \tag{3.41}
$$

and

$$
\varpi_n = \begin{bmatrix} X_{1,j_1-1} \\ X_{j_1,j_2-1} \\ X_{j_2,j_{3-1}} \\ \vdots \\ X_{j_{k-1},n} \end{bmatrix} \tag{3.42}
$$

in which

$$
X_{1,j_1-1} = \begin{bmatrix} 1 \otimes w_{1,j_1,j_2,\ldots,j_{k-1}} \\ 1 \otimes w_{2,j_1,j_2,\ldots,j_{k-1}} \\ \vdots \\ 1 \otimes w_{j_1-1,j_1,j_2,\ldots,j_{k-1}} \end{bmatrix}, \qquad X_{j_1,j_2-1} = \begin{bmatrix} 2 \otimes w_{j_1,j_1,j_2,\ldots,j_{k-1}} \\ 1 \otimes w_{j_1,j_1+1,j_2,\ldots,j_{k-1}} \\ \vdots \\ 1 \otimes w_{j_1,j_2-1,j_2,\ldots,j_{k-1}} \end{bmatrix}, \tag{3.43}
$$

$$
X_{j_2,j_{3-1}} = \begin{bmatrix} 2 \otimes w_{j_1,j_2,j_2,\ldots,j_{k-1}} \\ 1 \otimes w_{j_1,j_2,j_2+1,\ldots,j_{k-1}} \\ \vdots \\ 1 \otimes w_{j_1,j_2,j_3-1,\ldots,j_{k-1}} \end{bmatrix}, \quad \ldots, \quad X_{j_{k-1},n} = \begin{bmatrix} 2 \otimes w_{j_1,j_2,\ldots,j_{k-1},j_{k-1}} \\ 1 \otimes w_{j_1,j_2,\ldots,j_{k-1},j_{k-1}+1} \\ \vdots \\ 1 \otimes w_{j_1,j_2,\ldots,j_{k-1},n} \end{bmatrix}. \tag{3.44}
$$

We know that $N_{k-1}\varpi_n = 0$ has only zero solution if and only if $\mu_i + |\mu| - i + k \neq 0$, $\forall i \in I_1$. By Lemma 3.2.1 and the claim, the lemma is followed. $\square$

Next, we study the structure $(U(P)(1 \otimes_{\mathbb{F}} V))_{\langle j \rangle}$ as an $\bar{L}_n$-module and finally get a necessary and sufficient condition for the irreducibility of $L_{n+1}$-module $\mathcal{A} \otimes_{\mathbb{F}} V$ (cf. Lemma 3.2.3).

Note that $\bar{L}_n \simeq gl(n)$ (resp. $\bar{L}'_n \simeq sl(n)$ ) according to (3.5). Obviously, (3.12) implies that as $\bar{L}_n$-module $(\mathcal{A} \otimes_{\mathbb{F}} V)_{\langle j \rangle}$ is isomorphic to tensor product module $V(j\epsilon_1) \otimes_{\mathbb{F}} V$

$$(\mathcal{A}\otimes_{\mathbb{F}} V)_{\langle j\rangle} \cong V(j\varepsilon_1)\otimes_{\mathbb{F}} V(\mu) = \bigoplus_{\underline{c}\in I(\mu,j)} V(\mu+\underline{c}) \tag{3.45}$$

(cf. Lemma 2.1.2). Denote

$$p^{\underline{c}} = p_1^{c_1}p_2^{c_2}\cdots p_n^{c_n} \quad \text{for } \underline{c}=(c_1,\dots,c_n)\in\mathbb{N}^n. \tag{3.46}$$

Define the linear map

$$\begin{gathered}\varphi_j : (\mathcal{A}\otimes_{\mathbb{F}} V)_{\langle j\rangle} \to \big(U(P)(1\otimes_{\mathbb{F}} V)\big)_{\langle j\rangle},\\ \sum_{|\underline{l}|=j,\, q\in\overline{1,l}} a_{\underline{l},q}x^{\underline{l}}\otimes v_q \mapsto \sum_{|\underline{l}|=j,\, q\in\overline{1,l}} a_{\underline{l},q}p^{\underline{l}}.(1\otimes v_q).\end{gathered} \tag{3.47}$$

It is easy to verify that $\varphi_j$ is an $\bar{L}_n$-module homomorphism by (3.21).

For any $\underline{c}\in I(\mu,j)$ (cf. (2.11)), let $\xi_{\underline{c}}$ be a maximal vector for highest weight module $V(\mu+\underline{c})\subseteq(\mathcal{A}\otimes_{\mathbb{F}} V)_{\langle j\rangle}$ as $\bar{L}_n$-module. Then

$$\hat{\xi}_{\underline{c}} = \varphi_j(\xi_{\underline{c}}) \tag{3.48}$$

is also a maximal vector of $(U(P)(1\otimes_{\mathbb{F}} V))_{\langle j\rangle}$ with the same highest weight $\mu+\underline{c}$ as $\bar{L}_n$-module. Since the tensor product decomposition $V(j\varepsilon_1)\otimes_{\mathbb{F}} V(\mu)$ as $\bar{L}_n$-module is multiplicity-free and $(U(P)(1\otimes_{\mathbb{F}} V))_{\langle j\rangle}\subseteq(\mathcal{A}\otimes_{\mathbb{F}} V)_{\langle j\rangle}$, we can write

$$\hat{\xi}_{\underline{c}} = q_{\underline{c}}\xi_{\underline{c}} \tag{3.49}$$

for some $q_{\underline{c}}\in\mathbb{F}$. We know that

$$\big(U(P)(1\otimes_{\mathbb{F}} V)\big)_{\langle j\rangle}\cap V(\mu+\underline{c})\neq\{0\} \quad\Rightarrow\quad V(\mu+\underline{c})\subseteq\big(U(P)(1\otimes_{\mathbb{F}} V)\big)_{\langle j\rangle}. \tag{3.50}$$

So

$$\big(U(P)(1\otimes_{\mathbb{F}} V)\big)_{\langle j\rangle}\cap V(\mu+\underline{c})\neq\{0\} \quad\text{iff}\quad q_{\underline{c}}\neq 0. \tag{3.51}$$

In the following lemma, we will calculate $q_{\underline{c}}$ explicitly:

**Lemma 3.2.3.** *For any $\underline{c}\in I(\mu,j)$, we have*

$$q_{\underline{c}} = \prod_{s=1}^{n}\prod_{i=1}^{c_s}\big(\mu_s+|\mu|-s+i\big), \tag{3.52}$$

*where we treat $\prod_{i=1}^{c_s}(\mu_s+|\mu|-s+i)=1$ if $c_s=0$.*

**Proof.** Let $v_\mu$ be a highest weight vector of the given $gl(n)$-module $V$.

For any $\underline{c}\in I(\mu,j)$, we write a maximal vector $\xi_{\underline{c}}$ for highest weight module $V(\mu+\underline{c})$ as

$$\xi_{\underline{c}} = a_{\underline{c}}x^{\underline{c}}\otimes v_\mu + \sum_{(\nu,\underline{l})\neq(\mu,\underline{c}),\ \underline{c}+\mu=\underline{l}+\nu,\ i\in\overline{1,m(\nu)}} a^i_{\nu,\underline{l}}x^{\underline{l}}\otimes v^i_\nu, \quad 0\neq a_{\underline{c}},\ a^i_{\nu,\underline{l}}\in\mathbb{F}, \tag{3.53}$$

where $m(\nu)$ denotes the multiplicity of weight $\nu\in\Pi_\mu$.

**Case 1.** $c_n \neq 0$.

Note that $[x_s\partial_{x_t}, \partial_{x_n}] = 0$, $\forall 1 \leqslant s < t \leqslant n$. We know

$$0 \neq \partial_{x_n}.\xi_{\underline{c}} = a_{\underline{c}} c_n x^{\underline{c}-\varepsilon_n} \otimes v_\mu + \sum_{(\nu,\underline{l}) \neq (\mu,\underline{c}),\, \underline{c}+\mu=\underline{l}+\nu,\, i\in\overline{1,m(\nu)}} a^i_{\nu,\underline{l}} l_n x^{\underline{l}-\varepsilon_n} \otimes v^i_\nu \tag{3.54}$$

is a maximal vector for the $\bar{L}_n$-highest weight module $V(\mu + \underline{c} - \varepsilon_n)$. Set

$$\xi_{\underline{c}-\varepsilon_n} = \partial_{x_n}.\xi_{\underline{c}}. \tag{3.55}$$

Obviously,

$$\hat{\xi}_{\underline{c}} = a_{\underline{c}} p^{\underline{c}}.(1 \otimes v_\mu) + \sum_{(\nu,\underline{l}) \neq (\mu,\underline{c}),\, \underline{c}+\mu=\underline{l}+\nu,\, i\in\overline{1,m(\nu)}} a^i_{\nu,\underline{l}} p^{\underline{l}}.\big(1 \otimes v^i_\nu\big), \tag{3.56}$$

$$\begin{aligned} \hat{\xi}_{\underline{c}-\varepsilon_n} &= \varphi_j(\partial_{x_n}.\xi_{\underline{c}}) \\ &= a_{\underline{c}} c_n p^{\underline{c}-\varepsilon_n}.(1 \otimes v_\mu) + \sum_{(\nu,\underline{l}) \neq (\mu,\underline{c}),\, \underline{c}+\mu=\underline{l}+\nu,\, i\in\overline{1,m(\nu)}} a^i_{\nu,\underline{l}} l_n p^{\underline{l}-\varepsilon_n}.\big(1 \otimes v^i_\nu\big). \end{aligned} \tag{3.57}$$

Write

$$\partial_{x_n}.\hat{\xi}_{\underline{c}} = b_{\underline{c}}(n)\hat{\xi}_{\underline{c}-\varepsilon_n}, \quad b_{\underline{c}}(n) \in \mathbb{F}. \tag{3.58}$$

By (3.21) and (3.54), we have

$$\begin{aligned} \partial_{x_n}.\hat{\xi}_{\underline{c}} &= \partial_{x_n}.\varphi_j(\xi_{\underline{c}}) \\ &= c_n a_{\underline{c}} p^{\underline{c}-\varepsilon_n} \Delta^j_{n,n}(1 \otimes v_\mu) \\ &\quad + \sum_{(\nu,\underline{l}) \neq (\mu,\underline{c}),\, \underline{c}+\mu=\underline{l}+\nu,\, i\in\overline{1,m(\nu)}} a^i_{\nu,\underline{l}} \left[ \sum_{s=1}^{n-1} l_s p^{\underline{l}-\varepsilon_s}\big(1 \otimes E_{s,n} v^i_\nu\big) + l_n p^{\underline{l}-\varepsilon_n} \Delta^j_{n,n}\big(1 \otimes v^i_\nu\big) \right]. \end{aligned} \tag{3.59}$$

Denote

$$E_{s,n}.v^i_{\mu+\varepsilon_n-\varepsilon_s} = \Im_i v_\mu, \quad \Im_i \in \mathbb{F}. \tag{3.60}$$

Then, we have

$$b_{\underline{c}}(n) = \frac{c_n a_{\underline{c}}(j - 1 + \mu_n + |\mu|) + \sum_{s=1}^{n-1}(1 + c_s)\sum_{i=1}^{m(\mu-\varepsilon_s+\varepsilon_n)} a^i_{\mu-\varepsilon_s+\varepsilon_n,\underline{c}+\varepsilon_s-\varepsilon_n}\Im_i}{c_n a_{\underline{c}}} \tag{3.61}$$

by (3.58)–(3.61). For any $s \in \overline{1, n-1}$, we get

$$\begin{aligned}0 &= x_s \partial_{x_n}.\xi_{\underline{c}} \\ &= c_n a_{\underline{c}} x^{\underline{c}+\varepsilon_s-\varepsilon_n} \otimes v_\mu + \sum_{(\nu,\underline{l})\neq(\mu,\underline{c}),\, \underline{c}+\mu=\underline{l}+\nu,\, i\in\overline{1,m(\nu)}} a^i_{\nu,\underline{l}}\big[l_n x^{\underline{l}+\varepsilon_s-\varepsilon_n} \otimes v^i_\nu + x^{\underline{l}} \otimes E_{s,n}.v^i_\nu\big].\end{aligned} \tag{3.62}$$

Therefore,

$$c_n a_{\underline{c}} + \sum_{i=1}^{m(\mu-\varepsilon_s+\varepsilon_n)} a^i_{\mu-\varepsilon_s+\varepsilon_n,\underline{c}+\varepsilon_s-\varepsilon_n} \Im_i = 0. \tag{3.63}$$

Hence, we have

$$b_{\underline{c}}(n) = c_n - n + \mu_n + |\mu|. \tag{3.64}$$

Furthermore, by (3.55), we obtain

$$\partial_{x_n}.\hat{\xi}_{\underline{c}} = q_{\underline{c}} \partial_{x_n}.\xi_{\underline{c}} = q_{\underline{c}} \xi_{\underline{c}-\varepsilon_n} = b_{\underline{k}}(n) \hat{\xi}_{\underline{c}-\varepsilon_n} = b_{\underline{k}}(n) q_{\underline{c}-\varepsilon_n} \xi_{\underline{c}-\varepsilon_n} \tag{3.65}$$

which implies that

$$q_{\underline{c}} = b_{\underline{c}}(n) q_{\underline{c}-\varepsilon_n}. \tag{3.66}$$

**Case 2.** $c_n = 0$.

Suppose $c_{n-1} \neq 0$. We claim that $l_n = 0$ for any $\underline{l}$ of $a^i_{\nu,\underline{l}} \neq 0$ in (3.54). Indeed, we can write

$$(\nu_1 - \nu_2, \ldots, \nu_{n-1} - \nu_n) = \sum_{i=1}^{n-1} (\mu_i - \mu_{i+1}) \varpi_i - \sum_{i=1}^{n-1} k_i \alpha_i. \tag{3.67}$$

Since $\nu_n + l_n = \mu_n + c_n$, we have $l_n = \mu_n - \nu_n = \mu_n - (\mu_n + k_{n-1})$ by (2.9). Thus we get $l_n + k_{n-1} = 0$. So $l_n = 0$.

Hence,

$$x_{n-1} \partial_{x_n}.\partial_{x_{n-1}}.\xi_{\underline{c}} = -\partial_{x_n}.\xi_{\underline{c}} + \partial_{x_{n-1}}.x_{n-1} \partial_{x_n}.\xi_{\underline{c}} = 0. \tag{3.68}$$

This implies that $\partial_{x_{n-1}}.\xi_{\underline{c}}$ is a maximal vector for the highest weight $\bar{L}_n$-module $V(\mu + \underline{c} - \varepsilon_{n-1})$. Repeating the process from (3.56) to (3.64), we get

$$q_{\underline{c}} = \big(\mu_{n-1} + |\mu| - (n-1) + c_{n-1}\big) q_{\underline{c}-\varepsilon_{n-1}}. \tag{3.69}$$

Then induction implies that

$$q_{\underline{c}} = \prod_{s=1}^{n} \prod_{i=1}^{c_s} \big(\mu_s + |\mu| - s + i\big). \tag{3.70}$$

Thus the lemma is proved. □

By Lemma 3.2.3, we know that $(U(P)(1\otimes_{\mathbb{F}} V))_{\langle j\rangle} = (\mathcal{A}\otimes_{\mathbb{F}} V)_{\langle j\rangle}$ iff $q_{\underline{c}} \neq 0$ for any $\underline{c} \in I(\mu, j)$, $0 < j \in \mathbb{N}$. Thus we get the following sufficient and necessary condition for the irreducibility of $L_{n+1}$-module $\mathcal{A}\otimes_{\mathbb{F}} V$:

**Proposition 3.2.4.** *The $L_{n+1}$-module $\mathcal{A}\otimes_{\mathbb{F}} V$ is irreducible iff $q_{\underline{c}} \neq 0$ for any $\underline{c} \in I(\mu, j)$, $0 < j \in \mathbb{N}$.*

### 3.3. Jordan–Holder series for $L_{n+1}$-module $\mathcal{A}\otimes_{\mathbb{F}} V$

In this section, we will assume that $\mathcal{A}\otimes_{\mathbb{F}} V \neq U(P)(1\otimes_{\mathbb{F}} V)$ and prove the irreducibility of the quotient module $(\mathcal{A}\otimes_{\mathbb{F}} V)/U(P)(1\otimes_{\mathbb{F}} V)$.

First, we study the structure of $(\mathcal{A}\otimes_{\mathbb{F}} V)/U(P)(1\otimes_{\mathbb{F}} V)$ as an $\bar{L}_n$-module (cf. Lemmas 3.3.1 and 3.3.2). Recall the $\bar{L}_n$-module homomorphism $\varphi_j$ defined by (3.47). Denote $\mathrm{Ker}(\varphi_j)$ by $\mathcal{R}_{\langle j\rangle}$ and set

$$I(\mu, j)' = \left\{\underline{c} \in I(\mu, j) \;\middle|\; q_{\underline{c}} = 0\right\}. \tag{3.71}$$

According to Lemma 3.2.3, we have the following result:

**Lemma 3.3.1.** *For any $1 \leqslant j \in \mathbb{N}$, we have $\mathcal{R}_{\langle j\rangle} = \bigoplus_{\underline{c}\in I(\mu, j)'} V(\mu + \underline{c})$ and the following direct sum of submodules for $\bar{L}_n$: $(\mathcal{A}\otimes_{\mathbb{F}} V)_{\langle j\rangle} = (U(P)(1\otimes_{\mathbb{F}} V))_{\langle j\rangle} \oplus \mathcal{R}_{\langle j\rangle}$.*

Let $1 \leqslant k \in \mathbb{N}$. Suppose that

$$(\mathcal{A}\otimes_{\mathbb{F}} V)_{\langle i\rangle} = \big(U(P)(1\otimes V)\big)_{\langle i\rangle} \quad \text{for any } i \in \overline{0, k}, \tag{3.72}$$

but

$$(\mathcal{A}\otimes_{\mathbb{F}} V)_{\langle j\rangle} \neq \big(U(P)(1\otimes V)\big)_{\langle j\rangle} \quad \text{when } j \geqslant k+1. \tag{3.73}$$

Then by Lemma 3.3.1, as $\bar{L}_n$-modules,

$$(\mathcal{A}\otimes_{\mathbb{F}} V)/U(P)(1\otimes_{\mathbb{F}} V) \cong \bigoplus_{j=k+1}^{\infty} \mathcal{R}_{\langle j\rangle}. \tag{3.74}$$

**Lemma 3.3.2.** *Let $1 \leqslant k \in \mathbb{N}$ such that* (3.72) *and* (3.73) *hold. Then $\bar{L}_n$-module $\mathcal{R}_{\langle k+1\rangle}$ is irreducible.*

**Proof.** Since $(\mathcal{A}\otimes_{\mathbb{F}} V)_{\langle k\rangle} = (U(P)(1\otimes V))_{\langle k\rangle}$, Lemma 3.2.3 implies

$$q_{\underline{c}} \neq 0 \quad \forall \underline{c} \in I(\mu, k). \tag{3.75}$$

For convenience, we denote

$$q_s(\underline{c}) = \prod_{i=1}^{c_s} \big(\mu_s + |\mu| - s + i\big). \tag{3.76}$$

So

$$q_{\underline{c}} = \prod_{s=1}^{n} q_s(\underline{c}). \tag{3.77}$$

Assume $\underline{m} \in I(\mu, k+1)'$, i.e. $q_{\underline{m}} = 0$. Since the $gl(n)$-modules

$$\begin{aligned} V\big((k+1)\varepsilon_1\big) \otimes_{\mathbb{F}} V &\simeq (\mathcal{A} \otimes_{\mathbb{F}} V)_{\langle k+1\rangle} \subset V(\varepsilon_1) \otimes_{\mathbb{F}} (\mathcal{A} \otimes_{\mathbb{F}} V)_{\langle k\rangle} \\ &\simeq V(\varepsilon_1) \otimes_{\mathbb{F}} \big(V(k\varepsilon_1) \otimes_{\mathbb{F}} V\big) \end{aligned} \tag{3.78}$$

in the sense of monomorphism, we know that

$$\exists \underline{t} \in \mathbb{N}^n \text{ and } r \in \overline{1,n}, \quad \text{such that} \quad \underline{t} \in I(\mu, k) \text{ and } \underline{m} = \underline{t} + \varepsilon_r. \tag{3.79}$$

Therefore, $m_r = t_r + 1$. The fact $q_{\underline{m}} = 0$ implies that $q_r(\underline{m}) = 0$ because $q_s(\underline{m}) = q_s(\underline{t}) \neq 0$ for $s \neq r$. Then $q_r(\underline{t}) \neq 0$ and $q_r(\underline{m}) = 0 = q_r(\underline{t})(\mu_r + |\mu| - r + t_r + 1)$ imply that

$$\mu_r + |\mu| - r + t_r + 1 = 0 = \mu_r + |\mu| - r + m_r. \tag{3.80}$$

Obviously, $1 \leqslant m_r = t_r + 1 \leqslant |\underline{m}| = k + 1$ and $\underline{m} \in I(\mu, k+1)' \subset I(\mu, k+1)$ imply that

$$m_r \leqslant \mu_{r-1} - \mu_r \tag{3.81}$$

by (2.11). Assume $1 \leqslant m_r < k$. Then by (2.11) and (3.81), we know there exists some $\underline{l} \in \mathbb{N}^n$ satisfying $l_r = m_r$ and $\underline{l} \in I(\mu, k)$. So $q_r(\underline{l}) = q_r(\underline{m}) = 0$. Furthermore, $q_{\underline{l}} = 0$, which contradicts (3.75). Therefore, $m_r = k + 1$, i.e. $\underline{m} = (k+1)\epsilon_r$.

Suppose there exists another $(k+1)\varepsilon_s \in I(\mu, k+1)'$ but $s \neq r$. Then $0 = \mu_r + |\mu| - r + k + 1 = \mu_s + |\mu| - s + k + 1$. This is impossible, since $\mu_s \geqslant \mu_r$ whenever $s < r$. Thus we prove that $|I(\mu, k+1)'| = 1$, i.e. $\bar{L}_n$-module $\mathcal{R}_{\langle k+1\rangle}$ is irreducible. □

Set

$$I_s = \{1\} \cup \big\{ j \in \overline{2,n} \mid \mu_{j-1} - \mu_j \geqslant s \big\}. \tag{3.82}$$

By the above two lemmas, we can give the proof of (i) in the Main Theorem:

**Proposition 3.3.3.** *The vector space $\mathcal{A} \otimes_{\mathbb{F}} V$ is an irreducible $sl(n+1)$-module if and only if*

$$\forall 1 \leqslant s \in \mathbb{N}, \quad \mu_i + |\mu| - i + s \neq 0 \quad \textit{for any } i \in I_s. \tag{3.83}$$

**Proof.** Assume that there exist $1 \leqslant s \in \mathbb{N}$ and $i \in I_s$ satisfying $\mu_i + |\mu| - i + s = 0$. Then $V(\mu + s\varepsilon_i) \subseteq \mathcal{R}_{\langle s\rangle}$ by (2.11), (3.71), (3.82) and Lemma 3.3.1. Hence, $(\mathcal{A} \otimes_{\mathbb{F}} V)_{\langle s\rangle} \neq (U(P)(1 \otimes V))_{\langle s\rangle}$, i.e. $sl(n+1)$-module $\mathcal{A} \otimes_{\mathbb{F}} V$ is reducible.

Suppose that $sl(n+1)$-module $\mathcal{A} \otimes_{\mathbb{F}} V$ is reducible. Let $k$ satisfying (3.72) and (3.73). By Lemma 3.3.2, we have $V(\mu + (k+1)\epsilon_s) = \mathcal{R}_{\langle k+1\rangle}$ for some $s \in I_{k+1}$. So Lemma 3.2.3 implies $\mu_s + |\mu| - s + k + 1 = 0$. □

**Remark 3.3.4.** The condition (3.83) is equivalent to (1.7) and (1.8) given in the Main Theorem.

In the rest of this section, we study the relationship between any $\bar{L}_n$-module $\mathcal{R}_{\langle j\rangle}$ and $\bar{L}_n$-module $\mathcal{R}_{\langle j+1\rangle}$ for any $j \geqslant k+1$ based on the decomposition of tensor module and the projection operator techniques for $gl(n)$. (Recall Lemma 2.1.2 and projection operators appeared in Section 2.2.) And we finally prove the irreducibility of quotient module $(\mathcal{A} \otimes_{\mathbb{F}} V)/U(P)(1 \otimes_{\mathbb{F}} V)$ (cf. Proposition 3.3.8).

**Lemma 3.3.5.** *Let $k+1 \leqslant s \in \mathbb{N}$. For $\nu = \mu + \underline{l}$ with $\underline{l} \in I(\mu, s+1)'$, there exist $\nu' = \mu + \underline{m}$ with $\underline{m} \in I(\mu, s)'$ and $r \in \overline{1,n}$ such that $\nu = \nu' + \varepsilon_r$ and $\mu_r + |\mu| - r + m_r + 1 \neq 0$.*

**Proof.** Suppose $\nu = \mu + \underline{l}$ with $\underline{l} \in I(\mu, s+1)'$. Then $q_{\underline{l}} = 0$.

**Claim.** *There exists $r \in \overline{1,n}$ such that $\mu_r + |\mu| - r + l_r \neq 0$ and $l_r \geqslant 1$.*

Set

$$I_{\underline{l}} = \{t \in \overline{1,n} \mid \mu_t + |\mu| - t + l_t = 0\}. \tag{3.84}$$

It follows that $|I_{\underline{l}}| = 0, 1$. Otherwise, $\mu_t + |\mu| - t + l_t = 0 = \mu_q + |\mu| - q + l_q$ for some $t < q$. This is impossible because $\mu + \underline{l}$ is a highest weight implies that $l_t + \mu_t \geqslant l_q + \mu_q$ whenever $t < q$. It is obvious that the claim holds when $|I_{\underline{l}}| = 0$.

Now assume $|I_{\underline{l}}| = 1$ and $r_0 \in I_{\underline{l}}$. If the claim does not hold, then $\underline{l} = (s+1)\varepsilon_{r_0}$ and $\mu_t + |\mu| - t \neq 0$ for any $t \neq r_0$. From Lemma 3.3.2, we know $\mathcal{R}_{\langle k+1\rangle} = V(\mu + (k+1)\varepsilon_t)$ for some $t \in I_{k+1}$. Thus, Lemma 3.2.3 implies $q_{(k+1)\epsilon_t} = \prod_{i=1}^{k+1}(\mu_t + |\mu| - t + i) = 0$. Assume $\mu_t + |\mu| - t + r = 0$ for some $r \in \overline{1, k+1}$. On the other hand, $\mu_{r_0} + |\mu| - r_0 + s + 1 = 0$ by (3.84) due to $r_0 \in I_{\underline{l}}$. Hence, we have $\mu_{r_0} - \mu_t = r_0 + r - (t + s + 1)$. If $r_0 \leqslant t$, then $\mu_{r_0} - \mu_t \geqslant 0$; which contradicts $r_0 + r - (t+s+1) = r_0 - t + r - (s+1) < 0$. If $r_0 > t$, then $\mu_t + l_t = \mu_t \geqslant \mu_{r_0} + l_{r_0} = \mu_{r_0} + s + 1$ because $\mu + \underline{l} = \mu + (s+1)\varepsilon_{r_0}$ is a highest weight. Therefore, $\mu_t - \mu_{r_0} \geqslant s+1$; i.e. $t + s + 1 - (r_0 + r) \geqslant s + 1$. Hence, $r_0 - t + r \leqslant 0$. A contradiction arises. Thus the claim holds.

Suppose that $r$ satisfies the claim. Take $\nu' = \nu - \varepsilon_r$, $\underline{m} = \underline{l} - \varepsilon_r$. We claim that $\underline{m} \in I(\mu, s)'$. In fact, $\underline{m} \in I(\mu, s)$ by (2.11) because $l_r - 1 \leqslant s$ and $l_r \leqslant \mu_{r-1} - \mu_r$ implies $l_r - 1 < \mu_{r-1} - \mu_r$. Furthermore, $q_{\underline{l}} = (\mu_r + |\mu| - r + l_r)q_{\underline{m}} = 0$ implies that $q_{\underline{m}} = 0$. Therefore, $\underline{m} \in I(\mu, s)'$. Thus the lemma follows. □

**Lemma 3.3.6.** *We have $\partial_l(\mathcal{R}_{\langle j\rangle}) \nsubseteq (U(P)(1 \otimes V))_{\langle j-1\rangle}$ for any $l \in \overline{1,n}$ and $j > k+1$.*

**Proof.** Assume

$$\partial_l(\mathcal{R}_{\langle j\rangle}) \subseteq \big(U(P)(1 \otimes_{\mathbb{F}} V)\big)_{\langle j-1\rangle} \quad \text{for some } l \in \overline{1,n} \text{ and } j > k+1. \tag{3.85}$$

By (3.11), we know that

$$\partial_l\big(U(P)(1 \otimes_{\mathbb{F}} V)_{\langle j\rangle}\big) \subseteq \big(U(P)(1 \otimes_{\mathbb{F}} V)\big)_{\langle j-1\rangle}. \tag{3.86}$$

Then (3.85), (3.86) and Lemma 3.3.1 imply that

$$\partial_l\big((\mathcal{A} \otimes_{\mathbb{F}} V)_{\langle j\rangle}\big) \subseteq \big(U(P)(1 \otimes V)\big)_{\langle j-1\rangle}, \tag{3.87}$$

that is,

$$\partial_l\big((\mathcal{A} \otimes_{\mathbb{F}} V)_{\langle j\rangle}\big) \subsetneq (\mathcal{A} \otimes_{\mathbb{F}} V)_{\langle j-1\rangle}. \tag{3.88}$$

This contradicts the fact

$$\partial_l\big((\mathcal{A}\otimes_{\mathbb{F}} V)_{\langle j\rangle}\big)=(\mathcal{A}\otimes_{\mathbb{F}} V)_{\langle j-1\rangle}\quad\text{for any } l\in\overline{1,n}. \tag{3.89}$$

Thus the lemma follows. □

Let $\{e_1',\dots,e_n'\}$ be a basis for $\bar{L}_n$-module $V(\varepsilon_1)$. For any $1\leqslant j\in\mathbb{N}$, we define the following linear map:

$$\begin{aligned} &T_j: V(\varepsilon_1)\otimes_{\mathbb{F}}(\mathcal{A}\otimes_{\mathbb{F}} V)_{\langle j\rangle}\to(\mathcal{A}\otimes_{\mathbb{F}} V)_{\langle j+1\rangle},\\ &T_j\big(e_i'\otimes v\big)=p_i.v,\quad \forall v\in(\mathcal{A}\otimes_{\mathbb{F}} V)_{\langle j\rangle}. \end{aligned}\tag{3.90}$$

**Lemma 3.3.7.** *The linear map $T_j$ is an intertwining operator from the tensor module $V(\varepsilon_1)\otimes_{\mathbb{F}}(\mathcal{A}\otimes_{\mathbb{F}} V)_{\langle j\rangle}$ to $(\mathcal{A}\otimes_{\mathbb{F}} V)_{\langle j+1\rangle}$ for $\bar{L}_n$.*

**Proof.** For any $s,t,i\in\overline{1,n}$ and $v\in(\mathcal{A}\otimes_{\mathbb{F}} V)_{\langle j\rangle}$, we have

$$\begin{aligned} T_j\big(x_s\partial_{x_t}.\big(e_i'\otimes v\big)\big)&=T_j\big(x_s\partial_{x_t}\big(e_i'\big)\otimes v+e_i'\otimes x_s\partial_{x_t}.v\big)\\ &=T_j\big(\delta_{t,i}e_s'\otimes v+e_i'\otimes x_s\partial_{x_t}.v\big)=\delta_{t,i}p_s.v+p_i.x_s\partial_{x_t}.v=x_s\partial_{x_t}.p_i.v\\ &=x_s\partial_{x_t}.T_j\big(e_i'\otimes v\big) \end{aligned}\tag{3.91}$$

by (3.21). Thus the lemma follows. □

Based on Lemmas 3.3.5–3.3.7, we can prove (ii) of the Main Theorem in the following:

**Proposition 3.3.8.** *If $\mathcal{A}\otimes_{\mathbb{F}} V\neq U(P)(1\otimes_{\mathbb{F}} V)$, then $\{0\}\subset U(P)(1\otimes_{\mathbb{F}} V)\subset\mathcal{A}\otimes_{\mathbb{F}} V$ is a Jordan–Holder series for $L_{n+1}$-module $\mathcal{A}\otimes_{\mathbb{F}} V$.*

**Proof.** Suppose that $W+U(P)(1\otimes_{\mathbb{F}} V)$ is any nonzero submodule of quotient module $(\mathcal{A}\otimes_{\mathbb{F}} V)/U(P)(1\otimes_{\mathbb{F}} V)$, where $W=\bigoplus_{j\geqslant k+1}W\cap\mathcal{R}_{\langle j\rangle}$ is a weighted subspace of $\mathcal{A}\otimes_{\mathbb{F}} V$. By Lemma 3.3.6, we get $W\cap\mathcal{R}_{\langle k+1\rangle}\neq\{0\}$.

By Lemma 3.3.5, we know that for any $s\geqslant k+1$ and $\nu=\mu+\underline{l}$ with $\underline{l}\in I(\mu,s+1)'$, there exist $\nu'=\mu+\underline{m}$ with $\underline{m}\in I(\mu,s)'$ and $r\in\overline{1,n}$ such that $\nu=\nu'+\varepsilon_r$ and $\mu_r+|\mu|-r+m_r+1\neq 0$. Therefore, the highest weight module $V(\nu)(\subseteq\mathcal{R}_{\langle s+1\rangle})$ of highest weight $\nu$ appears in the decomposition of $\bar{L}_n$-tensor module $V(\varepsilon_1)\otimes_{\mathbb{F}} V(\nu')$ $(\subseteq V(\varepsilon_1)\otimes_{\mathbb{F}}\mathcal{R}_{\langle s\rangle}\subseteq V(\varepsilon_1)\otimes_{\mathbb{F}}(\mathcal{A}\otimes_{\mathbb{F}} V)_{\langle s\rangle})$.

**Claim.** *There exists some maximal vector $v_\nu$ (resp. $\xi_{\underline{m}+\varepsilon_r}$) of highest weight module $V(\nu)\subseteq V(\varepsilon_1)\otimes_{\mathbb{F}} V(\nu')$ (resp. $V(\nu)\subseteq\mathcal{R}_{\langle s+1\rangle}$) satisfying*

$$T_s(v_\nu)=\big(\mu_r+|\mu|-r+m_r+1\big)\xi_{\underline{m}+\varepsilon_r}\neq 0. \tag{3.92}$$

Assume that $w_\nu$ is a maximal vector of irreducible module $V(\nu)\subseteq V(\varepsilon_1)\otimes_{\mathbb{F}} V(\nu')$. Since $T_s: V(\varepsilon_1)\otimes_{\mathbb{F}}(\mathcal{A}\otimes_{\mathbb{F}} V)_{\langle s\rangle}\to(\mathcal{A}\otimes_{\mathbb{F}} V)_{\langle s+1\rangle}$ is an intertwining operator for $\bar{L}_n$, we know $T_s(w_\nu)$ is also a maximal vector of $\bar{L}_n$-module $V(\nu)\subseteq\mathcal{R}_{\langle s+1\rangle}$ $(\subseteq(\mathcal{A}\otimes_{\mathbb{F}} V)_{\langle s+1\rangle})$. Since the maximal vector $w_\nu$ must take the following form:

$$w_\nu=a_re_r'\otimes\varpi_{\nu'}+\sum_{j=1}^{r-1}\sum_{i=1}^{m(\nu'-\varepsilon_j+\varepsilon_r)}a_j^ie_j'\otimes v^i_{\nu'-\varepsilon_j+\varepsilon_r},\quad 0\neq a_r,\ a_j^i\in\mathbb{F}, \tag{3.93}$$

where $\varpi_{\nu'}$ is a maximal vector of the highest weight module $V(\nu')$ and the set $\{v^i_{\nu'-\varepsilon_j+\varepsilon_r} \mid i \in \overline{1, m(\nu'-\varepsilon_j+\varepsilon_r)}\}$ is a basis of the weight subspace $[V(\nu')]_{\nu'-\varepsilon_j+\varepsilon_r}$. For convenience, we write

$$v^i_{\nu'-\varepsilon_j+\varepsilon_r} = \sum_{\varepsilon_j-\varepsilon_r=\sum_{k=1}^m i_k(\varepsilon_{s_k}-\varepsilon_{t_k});\, s_i>t_i} a^i_{\underline{s},\underline{t}} (x_{s_1}\partial_{t_1})^{i_1}.\cdots.(x_{s_m}\partial_{t_m})^{i_m}.\varpi_{\nu'}. \tag{3.94}$$

Therefore,

$$T_s(w_\nu) = a_r p_r.\varpi_{\nu'} + \sum_{j=1}^{r-1} \sum_{i=1}^{m(\nu'-\varepsilon_j+\varepsilon_r)} a^i_j p_j.v^i_{\nu'-\varepsilon_j+\varepsilon_r}. \tag{3.95}$$

So (3.94) and (3.95) imply that

$$T_s(w_\nu) = n_{\varepsilon_r}.\varpi_{\nu'} \quad \text{for some } n_{\varepsilon_r} \in U(P \oplus \bar{L}_n). \tag{3.96}$$

Let $\varpi_{\nu'} = \xi_{\underline{m}}$ (resp. $\varpi_{\nu'} = \hat{\xi}_{\underline{m}}$) in (3.96), where

$$\xi_{\underline{m}} = a_{\underline{m}} x^{\underline{m}} \otimes v_\mu + \sum_{(\eta,\underline{l})\neq(\mu,\underline{m}),\, \underline{m}+\mu=\underline{l}+\eta,\, i\in\overline{1,m(\eta)}} a^i_{\eta,\underline{l}} x^{\underline{l}} \otimes v^i_\eta, \quad 0 \neq a_{\underline{m}},\ a^i_{\eta,\underline{l}} \in \mathbb{F}, \tag{3.97}$$

$$\hat{\xi}_{\underline{m}} = a_{\underline{m}} p^{\underline{m}}.(1 \otimes v_\mu) + \sum_{(\eta,\underline{l})\neq(\mu,\underline{m}),\, \underline{m}+\mu=\underline{l}+\eta,\, i\in\overline{1,m(\eta)}} a^i_{\eta,\underline{l}} p^{\underline{l}}.\big(1 \otimes v^i_\eta\big), \quad 0 \neq a_{\underline{m}},\ a^i_{\eta,\underline{l}} \in \mathbb{F}. \tag{3.98}$$

Take

$$\hat{\xi}_{\underline{m}+\varepsilon_r} = n_{\varepsilon_r}.\hat{\xi}_{\underline{m}}. \tag{3.99}$$

Then by Lemma 3.2.3, we have

$$\hat{\xi}_{\underline{m}+\varepsilon_r} = q_{\underline{m}+\varepsilon_r} \xi_{\underline{m}+\varepsilon_r} = n_{\varepsilon_r}.\hat{\xi}_{\underline{m}} = q_{\underline{m}} n_{\varepsilon_r}.\xi_{\underline{m}}. \tag{3.100}$$

Hence, we have

$$\big(\mu_r + |\mu| - r + m_r + 1\big)\xi_{\underline{m}+\varepsilon_r} = n_{\varepsilon_r}.\xi_{\underline{m}}. \tag{3.101}$$

Now we take

$$\begin{aligned} v_\nu = {} & \sum_{j=1}^{r-1} \sum_{i=1}^{m(\nu'-\varepsilon_j+\varepsilon_r)} \sum_{\varepsilon_j-\varepsilon_r=\sum_{k=1}^m i_k(\varepsilon_{s_k}-\varepsilon_{t_k});\, s_i>t_i} a^i_j a^i_{\underline{s},\underline{t}} e'_j \otimes (x_{s_1}\partial_{t_1})^{i_1}.\cdots.(x_{s_m}\partial_{t_m})^{i_m}.\xi_{\underline{m}} \\ & + a_r e'_r \otimes \xi_{\underline{m}}. \end{aligned} \tag{3.102}$$

Then $T_s(v_\nu) = (\mu_r + |\mu| - r + m_r + 1)\xi_{\underline{m}+\varepsilon_r} \neq 0$ by (3.101) and (3.102). Therefore, (2.30) and (3.92) imply that

$$\{0\} \neq (T_s|_{V(\varepsilon_1)\otimes_{\mathbb{F}} V(\nu')} \circ \tilde{P}_r)\big(V(\varepsilon_1) \otimes_{\mathbb{F}} V\big(\nu'\big)\big) = V(\nu), \tag{3.103}$$

where

$$\tilde{P}_r = \prod_{l \neq r} \left( \frac{\tilde{M} - \tilde{d}_l}{\tilde{d}_r - \tilde{d}_l} \right), \quad \tilde{d}_i = i - 1 - m_i - \mu_i, \tag{3.104}$$

and $\tilde{M}$ is the matrix in (2.27). Hence, $\mathcal{R}_{\langle s \rangle} \subseteq W$ for any $s \geqslant k+1$. Thus we prove $(\mathcal{A} \otimes_{\mathbb{F}} V)/U(P)(1 \otimes_{\mathbb{F}} V)$ is irreducible. □

Recall the notation $I_s$ in (3.82). Suppose $\mathcal{R}_{\langle k+1 \rangle} = V(\mu + (k+1)\varepsilon_r)$ for some $r \in I_{k+1}$. From the proof of Proposition 3.3.8, we know $(\mathcal{A} \otimes_{\mathbb{F}} V)/U(P)(1 \otimes_{\mathbb{F}} V)$ is generated by $\bar{\xi}_{\mu+(k+1)\varepsilon_r} = \xi_{\mu+(k+1)\varepsilon_r} + U(P)(1 \otimes_{\mathbb{F}} V)$, i.e. $(\mathcal{A} \otimes_{\mathbb{F}} V)/U(P)(1 \otimes_{\mathbb{F}} V) = U(L_{n+1}).\bar{\xi}_{\mu+(k+1)\varepsilon_r}$. Hence, we get the following result:

**Proposition 3.3.9.** *The $L_{n+1}$-module $(\mathcal{A} \otimes_{\mathbb{F}} V)/U(P)(1 \otimes_{\mathbb{F}} V)$ is isomorphic to the irreducible module $U(P)(1 \otimes_{\mathbb{F}} M)$, where $M$ is a gl$(n)$-irreducible module admitting the character $\chi_{\mu+(k+1)\varepsilon_r}$.*

**Proof.** Since $\partial_i.\xi_{\mu+(k+1)\varepsilon_r} \in (U(P)(1 \otimes V))_{\langle k \rangle} = (\mathcal{A} \otimes_{\mathbb{F}} V)_{\langle k \rangle}$, we get $\partial_i.\bar{\xi}_{\mu+(k+1)\varepsilon_r} = 0$ for any $i \in \overline{1,n}$. It follows from Proposition 3.3.8 that both $(\mathcal{A} \otimes_{\mathbb{F}} V)/U(P)(1 \otimes_{\mathbb{F}} V)$ and $U(P)(1 \otimes_{\mathbb{F}} M)$ are irreducible $L_{n+1}$-modules. So it is easy to verify that

$$\begin{aligned} \vartheta : (\mathcal{A} \otimes_{\mathbb{F}} V)/U(P)(1 \otimes_{\mathbb{F}} V) &\to U(P)(1 \otimes_{\mathbb{F}} M); \\ x.\bar{\xi}_{\mu+(k+1)\varepsilon_r} &\mapsto x.(1 \otimes v_{\mu+(k+1)\varepsilon_r}) \end{aligned} \tag{3.105}$$

is an $L_{n+1}$-module isomorphism, where $x \in U(P \oplus \bar{L}_n)$ and $v_{\mu+(k+1)\varepsilon_r}$ is a maximal vector of $M$. □

From Propositions 3.3.3, 3.3.8 and 3.3.9, we get the Main Theorem.

**Remark 3.3.10.** The irreducible module $U(P)(1 \otimes_{\mathbb{F}} V)$ is cyclic, i.e. it is generated by one vector. From Lemma 3.1.1 and (3.12), we know that $U(P)(1 \otimes_{\mathbb{F}} V)$ is in general not a highest weight module. We can easily verify the following result from the Main Theorem:

**Corollary 3.3.11.** *Assume one of the conditions in* (1.7) *and* (1.8) *fails. Denote*

$$i_0 = \min\{ i \in \overline{1,n} \mid \mu_i + |\mu| - i + 1 \in -\mathbb{N} \}. \tag{3.106}$$

*We have*:

(i) *The integer $k = -\mu_{i_0} - |\mu| + i_0 - 1$ satisfies* (3.72) *and* (3.73).

(ii) *The irreducible module $U(P)(1 \otimes_{\mathbb{F}} V)$ is finite-dimensional highest weight module with highest weight $k\omega_1 + \sum_{i=2}^{n} m_{i-1}\omega_i$ iff $i_0 = 1$, where $m_i = \mu_i - \mu_{i+1}$ for $i \in \overline{1, n-1}$.*

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